Coulomb corrections to the Delbrück scattering amplitude at low energies

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In this article, we study the Coulomb corrections to the Delbrück scattering amplitude. We consider the limit when the energy of the photon is much less than the electron mass. The calculations are carried out in the coordinate representation using the exact relativistic Green function of an electron in a Coulomb field. The resulting relative corrections are of the order of a few percent for scattering on for a large charge of the nucleus. We compare the corrections with the corresponding ones calculated through the dispersion integral of the pair production cross section and also with the magnetic loop contribution to the g-factor of a bound electron. The last one is in a good agreement with our results but the corrections calculated through the dispersion relation are not.

INTRODUCTION

The elastic scattering of photons by an external Coulomb field (the so called Delbrück scattering [1]) is one of nontrivial predictions of quantum electrodynamics. In the perturbation theory, the Delbrück scattering amplitude begins from the second order in $Z\alpha$ ($Z|e|$ is the charge of the nucleus, $\alpha = e^2 \approx 1/137$ is the fine-structure constant, we put $c = 1$, $\hbar = 1$). Therefore, significant efforts have been made to calculate the amplitude for the arbitrary scattering angles and energies even in the lowest-order Born approximation. The results of these calculations and the detailed bibliography can be found in the report [2].

To calculate the Delbrück scattering amplitude for $Z \gg 1$ it is necessary to take into account Coulomb field exactly. The expression for the amplitude exact in $Z\alpha$ has been derived in Ref. [3] without any additional assumptions, but numerical results have not yet been obtained because it is fairly cumbersome.

Considerable progress in the calculation of the Coulomb corrections to the lowest-order Born approximation has been achieved for the case of the photon energy ω much large than the electron mass m_e and either at small scattering angles $\Delta/\omega \ll 1$ ($\Delta = |\mathbf{k}_1 - \mathbf{k}_2|$, where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta of the photon in the initial and final states, correspondingly) Refs. [4, 5, 6, 7, 8] or at large momentum transfer $\Delta/m_e \gg 1$ Refs. [9, 10]. It turns out, that the Coulomb corrections strongly decrease the Delbrück amplitude in comparison with the lowest-order Born approximation (the theoretical results and the corresponding experimental data are reviewed in detail in the report [11] and the tables [12]). At the moment, the minimal photon energy at which Delbrück scattering is experimentally observed is $\omega = 889\,\mathrm{KeV}$ Refs. [13, 14, 15, 16, 17] but the corresponding energy for the Coulomb corrections is $\omega = 2754\,\mathrm{KeV}$ Ref. [18] only, i.e. above the electron-positron pair production threshold.

We aim here to calculate the Coulomb corrections at low energy $\omega \ll m_e$. These corrections have not yet been investigated neither experimentally nor theoretically. Nevertheless, they are closely connected with the Coulomb corrections to the pair production cross section due to the dispersion relation Refs. [19, 20] and, as we will show below, with the magnetic-loop contribution to the g-factor of a bound electron Ref. [21]. We think that the Delbrück scattering amplitude at low energy is useful to estimate the Coulomb corrections for both phenomena.

The amplitude is calculated in the coordinate representation with the help of the Green function of an electron moving in a Coulomb field. The structure of this paper is the following: in Section 2, we provide all necessary information about the Green function and the general parametrization of the Delbrück scattering amplitude. In Section 3, we show that the calculation in the coordinate representation reproduces the result of the lowest-order Born approximation derived in the momentum representation. We point out the difficulties specific for a calculation in the coordinate representation also occurring during the calculation of the Coulomb corrections. The results for the Coulomb corrections are given in Section 4. We also provide the simple parametrization of their dependence on Z. In Section 5 we compare our results with those obtained via the dispersion relation. The estimated value for the magnetic-loop contribution to the g-factor of a bound electron is given in Section 6.

DELBRÜCK SCATTERING AMPLITUDE

We parameterize the Delbrück scattering amplitude as follows:

$$A = \epsilon_\mu^{(1)}\, \epsilon_\nu^{\star(2)}\, \Pi^{\mu\nu}\left(\omega, \mathbf{k}_1, \mathbf{k}_2, Z\right), \tag{1}$$

$$
\begin{aligned}
\Pi^{\mu\nu}&\left(\omega, \mathbf{k}_1, \mathbf{k}_2, Z\right) = \\
&= \frac{\alpha\left(Z\alpha\right)^2}{m_e^3} \left\{ f_1\left(\omega, \mathbf{k}_1, \mathbf{k}_2, Z\right) \left[g^{\mu\nu}\, k_1 \cdot k_2 - k_2^\mu k_1^\nu\right]\right. \\
&\quad + f_2\left(\omega, \mathbf{k}_1, \mathbf{k}_2, Z\right) \left[\omega^2 g^{\mu\nu} - \omega\left(n^\mu k_1^\nu + n^\nu k_2^\mu\right)\right. \\
&\qquad\qquad\qquad \left.\left. + n^\mu n^\nu\, k_1 \cdot k_2\right]\right\} \tag{2}
\end{aligned}
$$

where $k_1 = (\omega, \mathbf{k}_1), k_2 = (\omega, \mathbf{k}_2)$ are the 4-momenta of the photon in the initial and final states, correspondingly, $\epsilon^{(1,2)}$ are the polarization vectors, the 4-vector n is defined as $k_1 \cdot n = k_2 \cdot n = \omega$, f_1 and f_2 are the form factors to be calculated. The main purpose of this article is to calculate $\lim_{|\mathbf{k}| \to 0} f_{1,2}(0, \mathbf{k}, \mathbf{k}, Z)$.

In a point-like charge approximation (Coulomb field), the polarization tensor $\Pi^{\mu\nu}$ has the following form:

$$\Pi^{\mu\nu}(\ldots, Z) = \tilde{\Pi}^{\mu\nu}(\ldots, Z) - \tilde{\Pi}^{\mu\nu}(\ldots, 0), \qquad (3)$$

$$\tilde{\Pi}^{\mu\nu}(\omega, \mathbf{k}_1, \mathbf{k}_2, Z) = i\alpha \int d^3 r_1 d^3 r_2 \exp(i\mathbf{k}_1\mathbf{r}_1 - i\mathbf{k}_2\mathbf{r}_2)$$

$$\times \int_C \frac{d\epsilon}{2\pi} \operatorname{Sp}\left\{ \gamma^\mu \hat{G}(\mathbf{r}_1, \mathbf{r}_2|\epsilon)\, \gamma^\nu G(\mathbf{r}_2, \mathbf{r}_1|\epsilon - \omega) \right\}, \qquad (4)$$

where $\hat{G}(\mathbf{r}_1, \mathbf{r}_2|\epsilon)$ is the Green function of an electron in a Coulomb field. The contour of integration over ϵ in the expression (4) goes from $-\infty$ to ∞ so that it is below the real axis in the left half-plane and above the real axis in the right half-plane. It is convenient to turn the contour along the imaginary axis. In this case the Green function takes the following form (see Ref. [22]):

$$G(\mathbf{r}_1, \mathbf{r}_2 \mid i\epsilon) = \frac{-1}{4\pi r_1 r_2\, p} \sum_{l=1}^{\infty} \int_0^\infty ds \exp(2iZ\alpha\, s\,(\epsilon/p) - p(r_1 + r_2)\coth s)$$

$$\times \left\{ \left[R_+ \frac{y}{2}\, I'_{2\nu}(y) B_l + R_- l I_{2\nu}(y) A_l \right] (\gamma_0\, i\epsilon + m) + Z\alpha\gamma^0 \left[im\,(\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2) + p\, R_+\, \coth s \right] I_{2\nu}(y)\, B_l \right.$$

$$\left. - i \left[p^2 \frac{r_1 - r_2}{2\sinh^2 s}\, (\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2)\, B_l + p \coth s\, (\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2)\, l A_l \right] I_{2\nu}(y) \right\}, \qquad (5)$$

where

$$R_\pm = 1 \pm \mathbf{n}_1 \mathbf{n}_2 \pm i\mathbf{\Sigma}\,(\mathbf{n}_1 \times \mathbf{n}_2), \qquad \Sigma^k = i\epsilon^{ijk}\left[\gamma^i, \gamma^j\right]/4, \qquad \hat{\mathbf{n}}_{(1,2)} = \boldsymbol{\gamma}\,\mathbf{n}_{(1,2)},$$

$$A_l = \frac{d}{dx}\left(P_l(x) + P_{l-1}(x)\right), \qquad B_l = \frac{d}{dx}\left(P_l(x) - P_{l-1}(x)\right), \qquad x = \mathbf{n}_1 \mathbf{n}_2,$$

$$\nu = \sqrt{l^2 - (Z\alpha)^2}, \qquad y = 2p\sqrt{r_1 r_2}/\sinh s, \qquad p = \sqrt{m^2 + \epsilon^2}. \qquad (6)$$

For the sake of convenience, we calculate the time-time component of the polarization tensor and the trace of the spatial components separately:

$$\Pi^{(ii)}(\mathbf{k}, Z) = (n_\mu n_\nu - g_{\mu\nu})\, \Pi^{\mu\nu}(0, \mathbf{k}, \mathbf{k}, Z), \qquad (7)$$

$$\Pi^{(00)}(\mathbf{k}, Z) = n_\mu n_\nu \Pi^{\mu\nu}(\mathbf{k}, Z). \qquad (8)$$

Substituting the parametrization of $\Pi^{\mu\nu}$ (2) in the right-hand side of the Eqs. (7) and (8) yields the relation between $\Pi^{(ii)}, \Pi^{(00)}$ and the form factors $f_{(1,2)}$:

$$\Pi^{(ii)}(\mathbf{k}, Z) = 2\frac{\alpha\,(Z\alpha)^2}{m_e^3}\, \mathbf{k}^2 f_1(0, \mathbf{k}, \mathbf{k}, Z) \qquad (9)$$

$$\Pi^{(00)}(\mathbf{k}, Z) = -\frac{\alpha(Z\alpha)^2}{m_e^3}\, \mathbf{k}^2\, (f_1(0, \mathbf{k}, \mathbf{k}, Z) + f_2(0, \mathbf{k}, \mathbf{k}, Z)). \qquad (10)$$
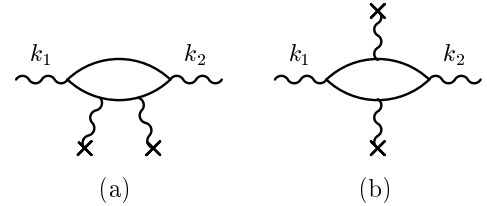

FIG. 1: lowest-order Born approximation

LOWEST-ORDER BORN APPROXIMATION

The diagrams of the second order of the perturbation theory in $Z\alpha$ are depicted in Fig. 1. Their contribution were calculated in Refs. [23, 24]. We aim here to demonstrate that the calculation of this diagrams in the coordinate representation reproduces the result of the lowest-order Born approximation.

To calculate the contributions of the diagrams, see Fig. 1, we expand the exponent function in the expres-

sion (4) up to the second order in $|\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}|$:

$$
\begin{aligned}
2\Pi^{\mu\nu}_{(a)} + \Pi^{\mu\nu}_{(b)} &= \frac{\alpha\,\mathbf{k}^2}{6} \int d^3r_1 d^3r_2 |\mathbf{r}_1 - \mathbf{r}_2|^2 \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \\
&\times \mathrm{Sp} \left\{ 2\,\gamma^\mu G^{(0)}\left(\mathbf{r}_1, \mathbf{r}_2 | i\epsilon\right) \gamma^\nu G^{(2)}\left(\mathbf{r}_2, \mathbf{r}_1 | i\epsilon\right) \right. \\
&\left. + \gamma^\mu G^{(1)}\left(\mathbf{r}_1, \mathbf{r}_2 | i\epsilon\right) \gamma^\nu G^{(1)}\left(\mathbf{r}_2, \mathbf{r}_1 | i\epsilon\right) \right\},
\end{aligned}
\tag{11}
$$

where $G^{(n)}$ is the contribution to the Green function (5)

of the n-th order in $Z\alpha$:

$$
\begin{aligned}
G^{(0)} &= - \left[(\gamma_0\, i\epsilon + m) + i\,\frac{\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\partial}{\partial |\mathbf{r}_1 - \mathbf{r}_2|} \right] \\
&\times \frac{\exp\left(-|\mathbf{r}_1 - \mathbf{r}_2|\right)}{4\pi |\mathbf{r}_1 - \mathbf{r}_2|},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
G^{(1)} = -\frac{Z\alpha}{16\pi} \frac{1 - t^2}{r_1 r_2} &\left\{ 2i\frac{\epsilon\rho}{p^2} \left[\gamma_0\, i\epsilon + m + \frac{ip^2}{\rho^2} (\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2) \frac{\partial}{\sigma\partial\sigma} + \frac{ip}{2} (\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2) \frac{\partial}{\partial\rho} \right] F(\rho, \sigma) \right. \\
&\left. + \frac{1}{1 - \sigma^2} \gamma^0 \left(\hat{R}_+ - i\frac{m}{p} (\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2) \frac{\partial}{\partial\rho} \right) \left(\frac{1}{\sigma} e^{-\rho\,\sigma} - e^{-\rho} \right) \right\},
\end{aligned}
\tag{13}
$$

$$
G^{(2)} = \frac{(Z\alpha)^2}{4\pi r_1 r_2} \int_0^\infty ds\, \exp(ip(r_1 + r_2)\coth s) \left\{ \sum_{l=1}^\infty \left[\left(\hat{R}_+ B_l \frac{y\partial}{2\,\partial y} + \hat{R}_- l A_l \right) (\gamma_0\, i\epsilon/p + m/p) \right. \right.
\tag{14}
$$

$$
\left. - i\frac{p(r_1 - r_2)}{2\sinh^2 s} (\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2) B_l - i\coth s\, (\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2)\, l A_l \right] \frac{\partial}{4l\,\partial\nu} I_{2\nu}(y) \Big|_{\nu=l}
\tag{15}
$$

$$
\left. + \frac{\epsilon s}{2p} \left[\frac{\epsilon s}{2p} \hat{T} + \gamma^0 \left(\frac{m}{p} (\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2) - \hat{R}_+ i\coth s \right) \frac{y}{\cos\frac{\phi}{2}} I_1 \left(y\cos\frac{\phi}{2} \right) \right] \right\},
\tag{16}
$$

where

$$
F(\rho, \sigma) = \frac{1}{2\rho\sigma} \left[e^{-\rho\,\sigma} \log\frac{1 + \sigma}{1 - \sigma} + e^{\rho\,\sigma} \Gamma\left(0, \rho(1 + \sigma)\right) - e^{-\rho\,\sigma} \Gamma\left(0, \rho(1 - \sigma)\right) \right],
\tag{17}
$$

$$
\hat{T} = \left[2\,y^2\, (\gamma_0\, i\epsilon/p + m/p) - i\,y^2 \coth s\, (\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2) - i\frac{p(r_1 - r_2)}{\sinh^2 s} \frac{(\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2)}{\cos^2\frac{\phi}{2}} \frac{y\partial}{\partial y} \right] I_0 \left(y\cos\frac{\phi}{2} \right),
\tag{18}
$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} \exp(-t)\, dt$ is the incomplete gamma function. The following notations have been also introduced in the expressions (12-18):

$$
\rho = p(r_1 + r_2), \qquad \sigma = \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{r_1 + r_2},
\tag{19}
$$

$$
t = \frac{r_1 - r_2}{r_1 + r_2}, \qquad \cos\frac{\phi}{2} = \left(\frac{1 + \mathbf{n}_1 \mathbf{n}_2}{2} \right)^{1/2}.
\tag{20}
$$

The contribution to the Green function $G^{(2)}$ of the second order is split into two parts: the contribution corresponding to the expansion of a Bessel function over an index (the part of the expressions (14), (15) separated by the square brackets) and the contribution in which we substitute the indices of the Bessel functions for $2l$ and sum over l explicitly. The latter can be called conditionally *quasiclassical* contribution because it corresponds to the contribution of the large angular momenta $l \gg Z\alpha$.

There are some points to be made. First of all, the contribution of each diagram Fig. 1 is infrared divergent, i. e. it is divergent at large distances. Since the divergence is

canceled between the diagrams Fig. 1(a) and Fig. 1(b), the contribution each of them depends on the regularization of this divergence. However, the contribution of separated terms (for example, the *quasiclassical* contribution or the contribution proportional to $\partial_\nu I_{2\nu}$ (14), (15)) and even the presence of the divergence depends on the order of the iterated integration over the spatial variables $\mathbf{r}_i$ and the inner variables of the Green functions – "proper times" s_i. An iterated improper integral, the value of which depends on the order of integration, we call a *conditionally convergent iterated* integral. The example of such integral, that arises during the calculation of the diagram Fig. 1(b), is given in Appendix. To avoid the complications due to an explicit regularization and difficulties during the calculation of separated terms, one should fix the order of integration for all the diagrams and sum the contribution of each one before the integration with respect to the last variable. We chose the variable t defined in (20) as the last integration variable. In this case, the contributions of the diagrams (Fig. 1)
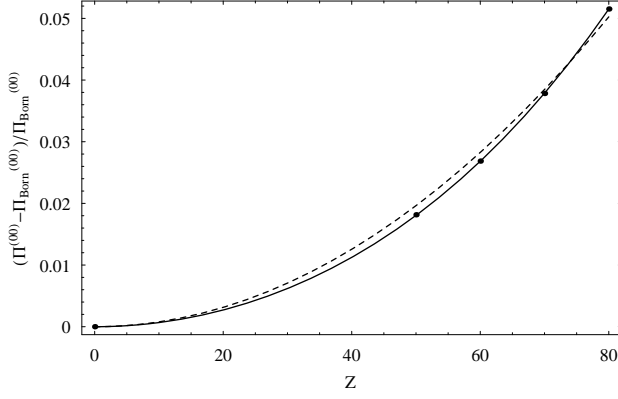
FIG. 2: The relative Coulomb corrections to the trace of the time-time component of the polarization tensor. The dashed curve corresponds to the fit $a(Z\alpha)^2$, the solid line corresponds to $a(Z\alpha)^2 + b(Z\alpha)^4$
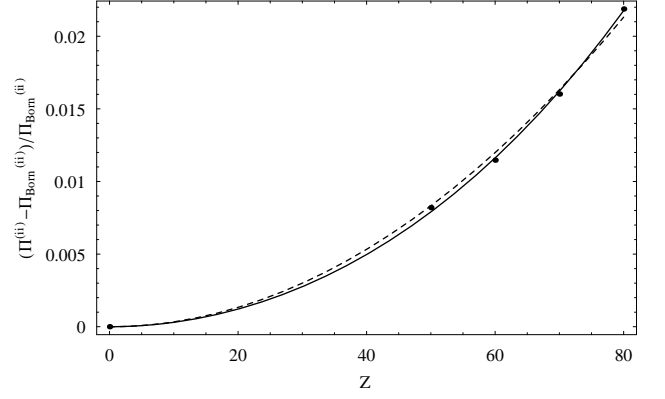

FIG. 3: The relative Coulomb corrections to the trace of the spatial components of the polarization tensor. The dashed curve corresponds to the fit $a(Z\alpha)^2$, the solid line corresponds to $a(Z\alpha)^2 + b(Z\alpha)^4$

have the following form:

$$\Pi_{(a)}^{(ii)} = \frac{\alpha(Z\alpha)^2}{m_e^3} \mathbf{k}^2 \left(-\frac{5}{2304} - \frac{5}{128} \int_0^1 \frac{dt}{t^2} \right), \qquad (21)$$

$$\Pi_{(b)}^{(ii)} = \frac{\alpha(Z\alpha)^2}{m_e^3} \mathbf{k}^2 \left(\frac{19}{1152} + \frac{5}{64} \int_0^1 \frac{dt}{t^2} \right), \qquad (22)$$

$$\Pi_{\text{Born}}^{(ii)} = 2\Pi_{(a)}^{(ii)} + \Pi_{(b)}^{(ii)} = \frac{\alpha(Z\alpha)^2}{m_e^3} \mathbf{k}^2 \frac{7}{576}. \qquad (23)$$

The time-time component is derived in a similar manner:

$$\Pi_{\text{Born}}^{(00)} = \frac{\alpha(Z\alpha)^2}{m_e^3} \mathbf{k}^2 \frac{59}{2304}. \qquad (24)$$

Substituting the expressions (23) and (24) in (9) and (10) yields the form factors (4) in the lowest-order Born approximation

$$f_{1\,\text{B}} = \frac{7}{16 \cdot 72}, \; f_{2\,\text{B}} = -\frac{73}{32 \cdot 72}. \qquad (25)$$

As noticed above, they coincide with the results derived in [23, 24].

COULOMB CORRECTIONS

Analytical derivation of the Coulomb corrections to the lowest-order Born approximation (25) is a rather complicated problem. We have calculated these corrections mostly numerically. To increase the accuracy of the numerical calculations we have subtracted the lowest-order Born approximation Eq. (11) from the general expression Eq. (4) before any transformations. Let us denote the angular momenta and the proper times that appear in the

Green functions in Eq. (4) as l_1, l_2 and s_1, s_2. After subtraction, it is convenient to change variables as follows: $(r_1, r_2) \rightarrow (\eta = \sqrt{r_1 r_2}, t' = r_1/r_2)$. Then we integrate analytically over ϵ, $x = (\mathbf{r}_1\mathbf{r}_2)/|\mathbf{r}_2||\mathbf{r}_1|$, t' one by one. After that we also perform the analytical summation over l_1. Thus, the expression (4) can be reduced to the sum over l_2 and the iterated integral over s_1, s_2 and η. The explicit expression for the integrand is omitted here as bulky. Further analytical integration is only possible for separate terms. The term previously referred to as the *quasiclassical* (which contains I_{2l} instead of $I_{2\nu}$) can be represented as an one-fold integral or as an infinite series over $Z\alpha$. For example, the corresponding contribution to $\Pi^{(00)}$ is the following :

$$\Pi_{\text{quasicl.}}^{(00)} - \Pi_{\text{quasicl.Born}}^{(00)} = \frac{\alpha(Z\alpha)^2}{m_e^4} \mathbf{k}^2$$

$$\times \left(\frac{287\pi^2 - 39}{18432}(Z\alpha)^2 - \frac{\pi^2(49\pi^2 - 15)}{69120}(Z\alpha)^4 \right.$$

$$+ \frac{\pi^4(158\pi^2 - 63)}{3096576}(Z\alpha)^6 - \frac{\pi^6(21\pi^2 - 10)}{5160960}(Z\alpha)^8$$

$$+ \left. \frac{\pi^8(83290\pi^2 - 46431)}{245248819200}(Z\alpha)^{10} + \dots \right). \qquad (26)$$

This contribution is finite, i. e. the infrared divergency is absent in Eq. (26). Nevertheless, the residual part must be integrated in the same order as that used to derive Eq. (26). We check explicitly that the contribution containing the subtraction from the single Bessel function, i. e. which is proportional to

$$I_{2\nu} - I_{2l} - \frac{(Z\alpha)^2}{2l} \partial_\nu I_{2\nu}\bigg|_{\nu=l} = O(Z^4\alpha^4), \qquad (27)$$

is a *conditionally convergent iterated* integral.

The results of our numerical calculations are presented in Figs. 2, 3 and Tab. I.

TABLE I: Relative Coulomb corrections

Z	$\left(\Pi^{(00)} - \Pi^{(00)}_{\text{Born}}\right)/\Pi^{(00)}_{\text{Born}}$	$\left(\Pi^{(ii)} - \Pi^{(ii)}_{\text{Born}}\right)/\Pi^{(ii)}_{\text{Born}}$
50	$1.82 \cdot 10^{-2}$	$8.20 \cdot 10^{-3}$
60	$2.69 \cdot 10^{-2}$	$1.15 \cdot 10^{-2}$
70	$3.78 \cdot 10^{-2}$	$1.60 \cdot 10^{-2}$
80	$5.15 \cdot 10^{-2}$	$2.19 \cdot 10^{-2}$

It should be noted that the *quasiclassical* contribution is of the order of 60% of the lowest-order Born approximation for $Z = 80$. The residual part containing the substraction from Bessel functions reduces the corrections up to a few percents. This cancellation adversely affects the accuracy of the calculation. The authors carried out the calculations of the corrections independently. The accuracy of the results (Tab. I) is better than one percent.

Now we consider the dependence of the Coulomb corrections on Z. The first two terms in Eq. (26) dominate and give $97 - 96\%$ of the *quasiclassical* contribution in spite of the fact that we calculate complete series in $Z\alpha$. It comes as a surprise that the complete results can be adequately fitted by a biquadratic polynomial in $Z\alpha$ without a free term:

$$\frac{\Pi^{(00)} - \Pi^{(00)}_{\text{Born}}}{\Pi^{(00)}_{\text{Born}}} = \frac{2304}{59}\left[(3.22 \pm 0.01) \cdot 10^{-3} (Z\alpha)^2 \right.$$
$$\left. + (1.90 \pm 0.02) \cdot 10^{-3} (Z\alpha)^4\right], \quad (28)$$

$$\frac{\Pi^{(ii)} - \Pi^{(ii)}_{\text{Born}}}{\Pi^{(ii)}_{\text{Born}}} = \frac{576}{7}\left[(6.69 \pm 0.17) \cdot 10^{-4} (Z\alpha)^2 \right.$$
$$\left. + (3.18 \pm 0.54) \cdot 10^{-4} (Z\alpha)^4\right]. \quad (29)$$

The results of the fitting with a quadratic function $a (Z\alpha)^2$ and also the functions in Eqs. (28), (29) are shown in Figs. 2, 3. One further comment is in order. The coefficients at $(Z\alpha)^2$ in Eqs. (28) and (29) have a magnitude one or two orders less than those at $(Z\alpha)^0$ in the lowest-order Born approximation Eqs. (23), (24). If one assumes the same hierarchy between the coefficients at $(Z\alpha)^2$ and $(Z\alpha)^4$, then the coefficients of $(Z\alpha)^4$ could not be distinguished from zero with our accuracy. In this case, the maximal difference between the dashed and solid curves in Figs. 2, 3 shows the actual accuracy of our calculations. Substituting Eqs. (28), (29) in the relations Eqs. (9), (10), we obtain the form factors $f_{(1,2)}$:

$$f_1 = \frac{7}{1152} + 3.35 \cdot 10^{-4}(Z\alpha)^2 + 1.6 \cdot 10^{-4}(Z\alpha)^4, \quad (30)$$

$$f_2 = -\frac{73}{2304} - 3.36 \cdot 10^{-3}(Z\alpha)^2$$
$$- 2.1 \cdot 10^{-3}(Z\alpha)^4. \quad (31)$$

PAIR PRODUCTION VIA DISPERSION RELATION

In the papers [20], [19], Gluckstern and Rohrlich have derived the relation between the pair production cross section in a Coulomb field and the Delbrück amplitude averaged over the polarizations:

$$A(\omega) = \frac{\omega^2}{2\pi^2}\int_{2m}^\infty \frac{\sigma_{\gamma\to e^+e^-}(\omega')}{\omega'^2 - \omega^2 + i0}\,d\omega'. \quad (32)$$

The amplitude (1) averaged over the polarizations of the photon has the form:

$$A = \frac{1}{2}\left(\delta^{ij} - \frac{k^i k^j}{\omega^2}\right)\Pi^{ij} = -\frac{\alpha (Z\alpha)^2}{m_e^3} f_2\,\omega^2. \quad (33)$$

One can find the relation between the Coulomb corrections to the form factor f_2 and the pair production cross section in a Coulomb field by using the expressions (32), (33) (we put $m_e = 1$ in this section):

$$f_2 = -\frac{1}{2\pi^2\alpha (Z\alpha)^2}\int_2^\infty \frac{\sigma(\omega')}{\omega'^2}\,d\omega'. \quad (34)$$

Let us check the formula (34) in the Born approximation. Substituting $Z = 82$ (lead) yields:

$$\frac{1}{2\pi^2\alpha (Z\alpha)^2 f_{2B}}\int_2^\infty \frac{-\sigma_B(\omega')}{\omega'^2}\,d\omega' = 1 + 4 \cdot 10^{-5}, \quad (35)$$

where σ_B is replaced by the asymptotical formulae derived by Maximon in Ref. [25] for $\omega < 2.1$:

$$\sigma_B(\omega) = \alpha (Z\alpha)^2 \frac{2\pi}{3}\left(\frac{\omega - 2}{\omega}\right)^3$$
$$\times\left(1 + \frac{\epsilon}{2} + \frac{23}{40}\epsilon^2 + \frac{11}{60}\epsilon^3 + \frac{29}{960}\epsilon^4 + \mathcal{O}(\epsilon^5)\right), \quad (36)$$
$$\epsilon = \frac{2\omega - 4}{2 + \omega + 2(2\omega)^{1/2}}, \quad (37)$$

for $\omega > 2.1$:

$$\sigma_B(\omega) = \alpha (Z\alpha)^2\left\{\frac{28}{9}\log 2\omega - \frac{218}{27}\right.$$
$$+ \left(\frac{2}{\omega}\right)^2\left[6\log 2\omega - \frac{7}{2} + \frac{2}{3}\log^3 2\omega - \log^2 2\omega\right.$$
$$\left. - \frac{\pi^2}{3}\log 2\omega + \frac{\pi^2}{6} + 2\zeta(3)\right] - \left(\frac{2}{\omega}\right)^4\left[\frac{3}{16}\log 2\omega + \frac{1}{8}\right]$$
$$\left. - \left(\frac{2}{\omega}\right)^6\left[\frac{29\log 2\omega}{9 \cdot 256} - \frac{77}{27 \cdot 512}\right] + \mathcal{O}\left(\frac{2^8}{\omega^8}\right)\right\}. \quad (38)$$

Now we discuss the Coulomb corrections to the form factors. Using the Eq. (31) we have obtained the relative correction to the form factor in the Born approximation
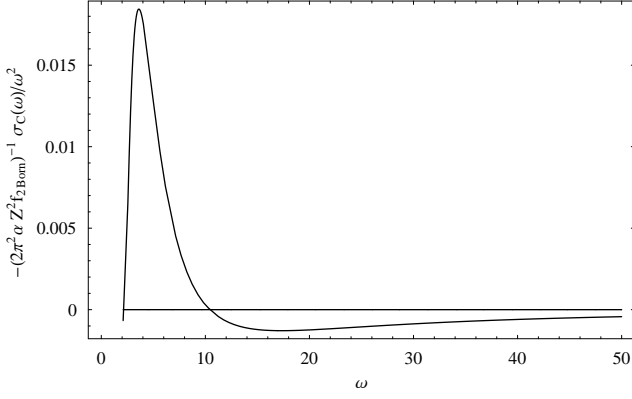
FIG. 4: Coulomb corrections to the pair production cross section ($Z = 82$)

(here and below all the calculations are carried out for $Z = 82$):

$$\frac{f_2 - f_{2B}}{f_{2B}} = 4.9 \cdot 10^{-2}. \qquad (39)$$

However, if we use the Coulomb corrections to the pair production cross section $\sigma_C(\omega) = \sigma(\omega) - \sigma_B(\omega)$ derived in Ref. [26] for the photon energy $\omega < 10$ and the interpolation equation derived in Ref. [27] for $\omega > 10$, then the relative correction to the form factor in the Born approximation is

$$-\frac{1}{2\pi^2\alpha(Z\alpha)^2 f_{2B}}\int_{2.01}^{\infty}\frac{\sigma_C(\omega)}{\omega^2}\,d\omega = 2.7 \cdot 10^{-3}. \quad (40)$$

This result is 20 times less than that in Eq. (39). The integrand (40) as a function of ω is shown in Fig. 4. It varies mainly in the region $2 < \omega < 30$ but there is a long negative "tail" for $\omega \to \infty$. The total integral is a result of the almost complete cancelation between the positive contribution for $\omega \lesssim 10$ and the negative one for $\omega \gtrsim 10$. The following ratio shows it clearly:

$$\frac{\int_2^{\infty}\sigma_T(\omega)/\omega^2\,d\omega}{\int_2^{\infty}|\sigma_T(\omega)/\omega^2|\,d\omega} = 3.9 \cdot 10^{-2}. \qquad (41)$$

For the integral (40) to be calculated with the sufficient accuracy it is necessary to derive the Coulomb corrections to the cross section with an accuracy better than a few percents. It is quite possible that this cancelation causes the discrepancy due to the lack of precision in the calculations of the positive part of the integrand in Ref.[26].

The cause of the discrepancy could also be the interpolation equation derived in Ref. [27] (the region $10 \lesssim \omega \lesssim 30$). Another interpolation formula for the Coulomb corrections to the pair production process is derived in Refs. [28, 29] up to terms which are of the order of $1/\omega$

TABLE II: Integration of the $1/\omega$-corrections over the negative "tail" [see Fig. 4 and Eq. (42)]

Contribution in σ_C when $\omega \to \infty$	$-\int_{\omega_0}^{\infty} d\omega \frac{\sigma_C(\omega)}{\omega^2\, 2\pi^2\alpha(Z\alpha)^2 f_{2B}}$
$\mathcal{O}(1)$	-0.184
$\mathcal{O}(1) + \mathcal{O}(1/\omega)$	0.068
$\mathcal{O}(1) + \mathcal{O}(1/\omega) + \mathcal{O}(1/\omega^2)$	-0.062

and $(1/\omega^2)\log\omega/2$:

$$\sigma_C(\omega \gg 2) = \alpha(\alpha Z)^2 \left\{ -\frac{28}{9}f(Z\alpha) + \frac{1}{\omega}\left[-\frac{\pi^4}{2}\,\mathrm{Im}\,g(Z\alpha) \right.\right.$$
$$\left.\left. -4\pi(Z\alpha)^3 f_1(Z\alpha)\right] + \frac{b}{\omega^2}\log\frac{\omega}{2} \right\}, \qquad (42)$$

where the functions f, g and f_1 are derived analytically but the coefficient b is obtained by an interpolation procedure from the experimental data for $\omega \gtrsim 30$ Refs. [30, 31]. The absolute value of the approximation formula (42) is always less than the corresponding corrections of Ref. [27] for $\omega > 25$. The expression (42) is zero when $\omega_0 = 8.95$ (see Fig. 4, the corresponding value for the approximation formula of Ref. [27] is $\omega_0 = 10.45$). In order to estimate the accuracy of the integral over the negative part, let us calculate the integral $-\int_{\omega_0}^{\infty} d\omega\,\sigma_C(\omega)/(\omega^2\, 2\pi^2\alpha(Z\alpha)^2 f_{2B})$ so that the terms of higher orders in $1/\omega$ are accounted for in $\sigma_C(\omega)$ one after another. The results are presented in the table (II). One can see that the successive terms from Eq. (42) thus integrated give the contributions of the same order, i. e. the process does not converge to a certain value of the integral.

It is also quite possible that, in order to resolve the contradiction between the results (39) and (40), the pair production in bound-free states should be taken into account because of the strong cancellation Eq. (41) of the contribution of free-free states.

The expression (40) coincides with that calculated in Ref. [32] (more precisely, $-D_1/f_{2B}$ in the notations of Ref. [32]). The comparison of our results, i. e. $(f_2(Z) - f_{2B})/f_{2B}$, and the results of Ref. [32] $-D_1/f_{2B}$ is made in Fig. 5.

It should be noted that our results and those of Ref. [32] are essentially different because the last one have a non-monotonic dependance on Z.

g-FACTOR OF A BOUND ELECTRON

The amplitude of virtual light-by-light scattering is known to be a part of the so-called magnetic loop contribution to the g-factor of a bound electron Ref. [34].
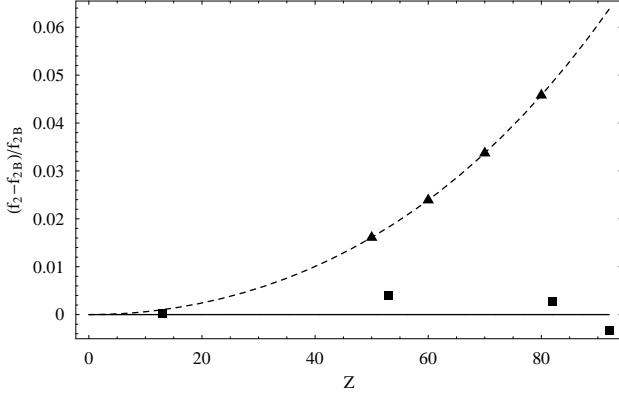
For the $1S_{1/2}$ electron state, this contribution reads

FIG. 5: Our results (triangles) for $(f_2(Z) - f_{2B})/f_{2B}$ and the approximation formula Eq. (31) (dashed line) in comparison with the results of Ref. [32] (squares)
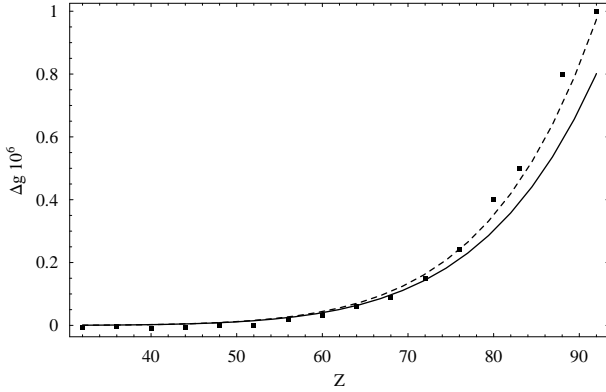
.

FIG. 6: The squares represent the difference $\Delta g_{\text{Ref.[21]}} - \Delta g_{\text{Ref.[33]}}$, the solid line corresponds to the function $(16/3) \, \alpha \, (Z\alpha)^5 \, (3.35 \cdot 10^{-4}(Z\alpha)^2)$, the dashed line is the function $(16/3) \, \alpha \, (Z\alpha)^5 \, (3.35 \cdot 10^{-4}(Z\alpha)^2 + 1.6 \cdot 10^{-4}(Z\alpha)^4)$, corresponding to the Coulomb corrections to the form factor f_1 Eq. (30).

(see Ref. [33]):

$$
\Delta g = -\frac{32}{3} \frac{\alpha(Z\alpha)^2}{\pi m^2} \int_0^\infty dq \, f_1 \left(q/m\right)
$$
$$
\times \int_0^\infty dr \, r \tilde{f}_1 \left(r\right) \tilde{f}_2 \left(r\right) \left(\frac{\sin qr}{qr} - \cos qr\right), \quad (43)
$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the radial parts of the electron wave function in a Coulomb field:

$$
\psi \left(\mathbf{r}\right) = \begin{pmatrix} \tilde{f}_1 \Omega \\ -i\tilde{f}_2 \left(\sigma \cdot \mathbf{n}\right) \Omega \end{pmatrix}, \quad (44)
$$

where Ω is the spherical spinor (see for example Ref. [35]). Using the lowest-order Born approximation for the form factor f_1 and the nonrelativistic expressions for the components of the wave function $\tilde{f}_1 \left(r\right) = 2 \exp(-r/a_B)$ and

$\tilde{f}_2 = \tilde{f}_1'(r)/2m$ yields the leading correction to the g-factor of a bound electron in $1S_{1/2}$ state Ref. [34]:

$$
\Delta g = \frac{7}{216} \, \alpha(Z\alpha)^5. \quad (45)
$$

In the case of small $Z\alpha$, one can expand the form factor f_1 in power series of $Z\alpha$:

$$
f_1 \left(0, 0, \mathbf{q}, Z\alpha\right) = F \left(\frac{q}{m}\right) + (Z\alpha)^2 \, F_{(1)} \left(\frac{q}{m}\right) + \mathcal{O} \left(Z^4\alpha^4\right). \quad (46)
$$

The contribution of $F \left(q/m\right)$ was considered in Ref. [33] in detail. To calculate the correction in $Z\alpha$, it is sufficient to use the functions $\tilde{f}_1$ and $\tilde{f}_2$ in the nonrelativistic limit and the expression (30) $f_1 - 7/1152$ as $F_{(1)}(0)$. The results of the numerical calculation of the magnetic-loop contribution exact in $Z\alpha$ are presented in Ref. [21].

The comparison of the contribution of the Coulomb corrections to the form factor f_1 Eq. (30) and the difference of the results obtained in Ref. [21] and Ref. [33] is depicted in Fig. 6. It is surprising that our correction coincides with this difference not only for the small $Z\alpha$, but for $Z\alpha \sim 1$ also.

CONCLUSIONS

The Coulomb corrections to the Delbrück scattering amplitude have been considered in this article. We have calculated these corrections in the low-energy limit but taking into account all orders of the parameter $Z\alpha$. The accuracy of the calculation is of the order of one percent for $Z = 50$ and increases with Z. Our result is in a good agreement with the corresponding contribution to the g-factor of a bound electron calculated previously in Ref. [21]. However, there is a contradiction with the dispersion integral of the Coulomb corrections to the pair production cross section calculated in Ref. [32].

EXAMPLE OF REARRANGEMENT OF A CONDITIONALLY CONVERGENT INTEGRAL

While calculating the contribution of the first order in $Z\alpha$, we have expanded the expression (5) on $Z\alpha$ and integrated over s. The equation (22) corresponding to the contribution of the diagram [Fig. 1(b)] has been derived by integration over ρ, σ and t in order [see Eq. (20)]. However, one can calculate this contribution in an alternative way – by substituting the expansion of Eq. (5) in Eq. (11) and integrating over r_1 and r_2 before the integration over s_1 and s_2 in the Green functions (5). One

of the typical expressions appeared is

$$Y(t_1, t_2, z) = \frac{1 + t_1 t_2}{(1 + 2z(t_1 + t_2)^2)^2 (t_1 + t_2)}$$
$$\times \left(1 - \frac{6(1 + t_1 t_2)}{(t_1 + t_2)^2}\right), \quad (A.47)$$

where $t_{1,2} = \coth s_{1,2} \in (1, \infty)$ and $z = (1+x)/2 \in (0,1)$. The expression (A.47) has to be integrated over the total variables' domain. One can easily integrate over t_1, z and t_2 one after another (or z, t_1 and t_2) and obtain a finite result, that is

$$\int_1^\infty \int_0^1 \int_1^\infty Y \, dt_1 \, dz \, dt_2 = \int_1^\infty \int_1^\infty \int_0^1 Y \, dz \, dt_1 \, dt_2$$
$$= \frac{133}{60} - \frac{13\pi}{4\sqrt{2}} + \frac{119 \arctan 2\sqrt{2}}{60\sqrt{2}} + \frac{38}{15} \log \frac{32}{81}. \quad (A.48)$$

However, if one integrates Eq. (A.47) over t_1 and t_2 at first then the result is the function of z

$$\tilde{Y}(z) = \int_1^\infty \left(\int_1^\infty Y(t_1, t_2, z) \, dt_1\right) dt_2$$
$$= \frac{1}{60z} \left\{ \frac{1 + 120z}{2\sqrt{2z}} \left(\arctan\left(2\sqrt{2z}\right) - \frac{\pi}{2}\right) \right.$$
$$\left. + 16z(5 - 48z) \log\left(\frac{1 + 8z}{8z}\right) + 96z - 1 \right\}, \quad (A.49)$$

which has a nonintegrable singularity at $z = 0$

$$\tilde{Y}(z) = -\frac{\pi}{240\sqrt{2} z^{3/2}} + O\left(z^{-1/2}\right). \quad (A.50)$$

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